Geometry and Topology Preliminary Examination
Northwestern University
Fall 2015

Answer all of the questions.

1. The real projective plane $\mathbb{RP}^2$ is the quotient space obtained by identifying antipodes $x \sim -x$ in the unit sphere in $\mathbb{R}^3$. Find a simplicial complex homeomorphic to the real projective plane.

Recall that a simplicial complex is a union of simplices such that each each face of a simplex is again a simplex, the $n+1$ vertices of any n-simplex are distinct, and if two simplices have the same set of vertices, they are equal.

2. Calculate the homology and cohomology groups with $\mathbb{Z}$ coefficients of the real projective plane $\mathbb{RP}^2$.

3. Let $n > 0$. Consider the standard covering of $\mathbb{RP}^n$ by open sets $\{U_\alpha\}_{0 \leq \alpha \leq n}$, where

$$U_\alpha = \{[x_0 : \ldots : x_n] \in \mathbb{RP}^n \mid x_\alpha \neq 0\},$$

and for each $d \in \mathbb{Z}$ define a line bundle E_d on $\mathbb{RP}^n$ by the transition functions

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{R}\backslash\{0\}, \quad g_{\alpha\beta} = \left(\frac{x_\beta}{x_\alpha}\right)^d.$$

Let now $n = 1$ and let $L \to \mathbb{RP}^1 \cong S^1$ be any line bundle. Since each U_α is contractible, L can be trivialized over U_α, and so L is given by transition functions

$$h_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{R}^*.$$

a) Show that if the image of all $h_{\alpha\beta}$ lies in $\mathbb{R}_{>0}$, for $\alpha \neq \beta$, then L is isomorphic to the trivial bundle.

b) Show that if the image of all $h_{\alpha\beta}$ lies in $\mathbb{R}_{<0}$ for $\alpha \neq \beta$, then L is isomorphic to the trivial bundle.

c) Show that if neither of the two previous situations occur, then $L \otimes E_1$ is isomorphic to the trivial bundle.

d) Conclude that the set of isomorphism classes of line bundles over $\mathbb{RP}^1$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. You may use the fact that E_1 is not isomorphic to the trivial bundle.

4. Consider Poincaré disk

$$M = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$$

with Riemannian metric

$$g = 4\frac{(dx)^2 + (dy)^2}{(1 - r^2)^2},$$

where $r = \sqrt{x^2 + y^2}$.

a) Rewrite this metric in polar coordinates.

b) Recall that the Christoffel symbol of the second kind Γ_{ij}^k is given by the formula

$$\Gamma_{ij}^k = g^{k\ell}\Gamma_{ij\ell},$$

where $\Gamma_{ij\ell}$, the Christoffel symbol of the first kind, is given by the formula

$$\frac{1}{2}(\partial_j g_{i\ell} + \partial_i g_{j\ell} - \partial_\ell g_{ij}).$$

Calculate the Christoffel symbol Γ_{rr}^r

5. Let $M = \mathbb{R}^2/\mathbb{Z}^2$ be the two dimensional torus, let $\pi : \mathbb{R}^2 \to M$ be the quotient map, let ℓ be the line in $\mathbb{R}^2$ given by the equation

$$3x = 7y,$$

and let $S = \pi(\ell) \subset M$.

a) Show that S is a compact embedded submanifold of M.

b) Find a closed differential 1-form α on M such that its cohomology class $[\alpha] \in H^1(M)$ is Poincaré dual to S.

6. The Brouwer fixed-point theorem states that every continuous map from the n-dimensional ball to itself has a fixed point. Prove this theorem as a corollary of the theorem that the homology group $H_{n-1}(S^{n-1})$ is nonzero.